NATURAL NUMBERS

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Abstract. We begin our study of number systems. The first section is a sketch of the development of the natural numbers, which gives us the principle of induction.

1. Natural Numbers

We wish to create a set which is allows us to count in a more or less formal way. The numbers we use to count are be labeled 0, 1, 2, et cetera, defined in a manner which reflects what we memorized as infants.

Having built the language of sets, we start with the simplist set, which is the empty set, and call it 0. Now 1 is naturally thought of as a set containing one element, and the most obvious choice for an this element is 0. Proceeding in this way, we would obtain

- $0 = \varnothing$;
- $1 = \{\varnothing\}$;
- $2 = \{\varnothing, \{\varnothing\}\}$;
- $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$;

and so forth. We could have written this as

- $0 = \varnothing$;
- $1 = \{0\}$;
- $2 = \{0, 1\}$;
- $3 = \{0, 1, 2\}$;

and so forth. Under this interpretation, a given natural number should be the set containing all of the previous natural numbers. Having made a plan for defining natural numbers, we proceed to attempt to formalize it.

We define 0 to be the empty set. If x is a set, the *successor* of x is denoted x^+ and is defined as

$$x^+ = x \cup \{x\}.$$

The *natural numbers* are the set $\mathbb{N}$ defined by following properties:

(1) $0 \in \mathbb{N}$;
(2) if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$;
(3) if $S \subset \mathbb{N}$, $0 \in S$, and $n \in S \Rightarrow n^+ \in S$, then $S = \mathbb{N}$.

2. Induction

Note that the third property of natural numbers asserts that only successors of 0 are in $\mathbb{N}$; that is, this property asserts that $\mathbb{N}$ is a minimal set of successors of 0, and that $\mathbb{N}$ is the unique set satisfying (1) through (3). This property is known as the *Principal of Mathematical Induction.*

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Suppose that for every natural number n, we have a proposition $p(n)$ which is either true or false. Let

$$S = \{n \in \mathbb{N} \mid p(n) \text{ is true}\}.$$

Now if $p(0)$ is true, and if the truth of $p(n)$ implies the truth of $p(n^+)$, then the set S contains 0 and it contains the successor of every element in it. Thus, in this case, $S = \mathbb{N}$, which means that $p(n)$ is true for all $n \in \mathbb{N}$. We state this as

Theorem 1. Induction Theorem
Let $p(n)$ be a proposition for each $n \in \mathbb{N}$. If

(1) $p(0)$ *is true;*
(2) *If $p(n)$ is true, then $p(n^+)$ is true;*

then $p(n)$ is true for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$, we say the m is less than or equal to n if $m \subset n$:

$$m \leq n \Leftrightarrow m \subset n.$$

Now the induction theorem can be made stronger by weakening the hypothesis. The resulting theorem gives a proof technique which is known as strong induction.

Theorem 2. Strong Induction Theorem
Let $p(n)$ be a proposition for each $n \in \mathbb{N}$. If

(1) $p(0)$ *is true;*
(2) *If $p(m)$ is true for all $m \leq n$, then $p(n+1)$ is true;*

then $p(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $t(n)$ be the statement that "p(m) is true for all $m \leq n$".

Our first assumption is that $p(0)$ is true, and since the only natural number less than or equal to 0 is zero (because the only subset of the empty set is itself), this means that $t(0)$ is true.

Our second assumption is that if $t(n)$ is true, then $p(n+1)$ is true. Thus assume that $t(n)$ is true so that $p(n+1)$ is also true. Then $p(i)$ is true for all $i \leq n+1$. Thus $t(n+1)$ is true.

By our original Induction Theorem, we conclude that $t(n)$ is true for all $n \in \mathbb{N}$. This implies that $p(n)$ is true for all $n \in \mathbb{N}$. $\square$

3. Recursion

We now state the Recursion Theorem, which will allows us to define addition and multiplication of natural numbers. It is possible to prove this theorem using strong induction.

Theorem 3. Recursion Theorem
Let X be a set, $f : X \to X$, and $a \in X$. Then there exists a unique function $\phi : \mathbb{N} \to X$ such that $\phi(0) = a$ and $\phi(n^+) = f(\phi(n))$ for all $n \in \mathbb{N}$.

Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = n^+$. Let $\sigma_m : \mathbb{N} \to \mathbb{N}$ be the unique function, whose existence is guaranteed by the Recursion Theorem, defined by $\sigma_m(0) = m$ and $\sigma_m(n^+) = f(\sigma_m(n)) = (\sigma_m(n))^+$. Then $\sigma_m(n)$ is defined to be the *sum* of m and n:

$$m + n = \sigma_m(n).$$

Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f = \sigma_m$. Let $\mu_m : \mathbb{N} \to \mathbb{N}$ be the unique function, whose existence is guaranteed by the Recursion Theorem, defined by $\mu_m(0) = 0$ and $\mu_m(n^+) = f(\mu_m(n)) = \sigma_m(\mu_m(n)) = m + \mu_m(n)$. Then $\mu_m(n)$ is defined to be the *product* of m and n:

$$mn = \mu_m(n).$$

The following properties of natural numbers can be proved using the above definitions:

- $m + n = n + m$ (commutativity of addition);
- $(m + n) + o = m + (n + o)$ (associativity of addition);
- $mn = nm$ (commutativity of multiplication);
- $(mn)o = m(no)$ (associativity of multiplication);
- $m(n + o) = mn + mo$ (distributivity of multiplication over addition);
- $m + 0 = m$ (0 is an additive identity);
- $1m = m$ (1 is a multiplicative identity);
- $0m = 0$.

We state two additional properties, which we will use to show that multiplication of integers is well-defined.

Proposition 1. Cancellation Law of Addition
Let $a, b, c \in \mathbb{N}$ and suppose that $a + c = b + c$. Then $a = b$.

Proposition 2. Cancellation Law of Multiplication
Let $a, b, c \in \mathbb{N}$ and suppose that $ac = bc$. Then $a = b$.

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